Lecture 3: Fundamental Theorem of Algebra and Vector Geometry

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1.3 The Fundamental Theorem of Algebra

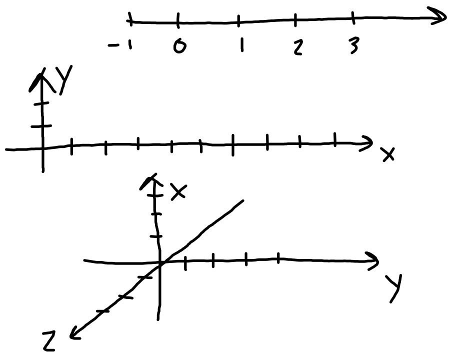
Gauss completed the first proof for this theorem which states:
Every polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z_1 + a_0 z_0$ with complex coefficients $a_n, a_{n-1}, a,\ a \in \mathbb{C}$, can be factored using complex numbers.
$p(z) = a_n * (z - w_1) * \cdots * (z - w_n)$

roots

2 Vector Geometry

2.1 Introduction

Visualizing vectors in different dimensions:

algebraically	geometrically
$\mathbb{R}$ ("scalars") $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$ $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$	
We can keep going like this... $\mathbb{R}^4 = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{R}\}$	eg. space-time

Therefore, for any number, *n*, in the natural number set:
$\mathbb{R}^n = \{(x_1, \quad , x_n) \mid x \in \mathbb{R}\}$

2.2 Algebraic structure of $\mathbb{R}^n$:

(1) $(x_1, \quad , x_n) = (y_1, \quad , y_n)$ $\overline{\text{or}}$ $x_i = y_i$ when $i \in \{1, \quad , n\}$

(2) $(x_1, \quad , x_n) + (y_1, \quad , y_n) = (x_1 + y_1, \quad , x_n + y_n)$

(3) **zero vector**: 0 = (0,…,0) $\in \mathbb{R}^n$

(4) negative if $x = \{x_1, \quad , x_n\}$ then $-x = (-x_1, \quad , -x_n)$

(5) multiplication by a scalar Let $k \in \mathbb{R}$
$k(x_1, \quad , x_n) = (kx_1, \quad , kx_n)$

2.3 Linear Combinations

Definition: Let $k_1, \ldots, k_m \in \mathbb{R}$ and $u_1, \ldots, u_m \in \mathbb{R}^n$. Then, the vector $k_1u_1 + k_2u_2 + k_3u_3 + \ldots + k_mu_m \in \mathbb{R}^n$ is called a linear combination of $u_1, \ldots, u_m$

Example:

u_1=(1,2,3)	u_2=(1,0,0)
k_1=1	k_2=7

Hence,
1*(1,2,3) + 7*(1,0,0) = (8,2,3)   is a linear combination of u_1 and u_2.

Observe that (0,1,0) is not a linear combination of u_1 and u_2:
(0,1,0) = k_1(1,2,3) + k_2(1,0,0)
0=k_1 + k_2
1=2k_1
0=3k_1
When solving for k_1 and k_2, the second and third equations contradict each other, and so (0,1,0) is not a linear combination of u_1 and u_2.

2.4 Properties of vector addition and scalar multiplication

V1 Associativity
u+(v+w)=(u+v)+w

"there exists"
V2 $\exists$ adding identity ($0 \in \mathbb{R}^n$)
that is: v+0=v

for all
V3 $\forall$ $v \in \mathbb{R}^n$ $\exists$ inverse $-v \in \mathbb{R}^n$
that is: v+(-v)=0

V4 Commutativity
v+w=w+v

S1 Distributivity I
k*(v+w)=k*v + k*w

S2 Distributivity II
(k+m)*v=kv + mv

S3 Associativity
k*(m*v)=(k*m)*v

S4 $1 \in \mathbb{R}$ does nothing
that is: 1v=v

for: u, v, w $\in \mathbb{R}^n$ and k, m $\in \mathbb{R}$

These rules are also known as the **vector spaces axioms**.

2.5 The Dot Product

Define two arbitrary vectors in 3D, and the dot product is equivalent to u·v

$u=(x_1,x_2,x_3)$ $v=(y_1,y_2,y_3)$

$$u\ v = x_1y_1 + x_2y_2 + x_3y_3 \in \mathbb{R}$$

Now that we have the dot product, we can define:

$\|u\| = \sqrt{u\ u} = \sqrt{x_1^2+x_2^2+x_3^2}$
length/norm of u

$\|u-v\|$ = distance between u and v

u and v are perpendicular, or orthogonal, if and only if their dot product equals 0.
$u\ v=0$

The above example is the dot product in the third dimension. We can expand this to the *n*th dimension below:

$u=(x_1,\ \ ,x_n)$ $v=(y_1,\ \ ,y_n)$

$u\ v = x_1y_1 + \ \ + x_ny_n$ $\in \mathbb{R}$
the dot product is always a real number